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## A numerical test of the continuum index theorem on the lattice

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### Abstract

The overlap formalism of chiral fermions provides a tool to measure the index,  $Q$ , of the chiral Dirac operator in a fixed gauge field background on the lattice. This enables a numerical measurement of the probability distribution,  $p(Q)$ , in Yang-Mills theories. We have obtained an estimate for  $p(Q)$  in pure SU(2) gauge theory by measuring  $Q$  on 140 independent gauge field configurations generated on a  $12^4$  lattice using the standard single plaquette Wilson action at a coupling of  $\beta = 2.4$ . This distribution is in good agreement with a recent measurement [8] of the distribution of the topological charge on the same lattice using the same coupling and the same lattice gauge action. In particular we find  $\langle Q^2 \rangle = 3.3(4)$  to be compared with  $\langle Q^2 \rangle = 3.9(5)$  found in [8]. The good agreement between the two distributions is an indication that the continuum index theorem can be carried over in a probabilistic sense on to the lattice.

## I. Introduction

The four dimensional version of the Atiyah-Singer index theorem [1] states that the index of the continuum Euclidean chiral Dirac operator,

$$C(A) = \sigma_\mu(\partial_\mu + iA_\mu(x)), \quad (1.1)$$

in a fixed gauge field background,  $A_\mu(x)$ , is equal to the topological charge,

$$Q = \frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\rho\sigma} \text{Tr } F_{\mu\nu}(x)F_{\rho\sigma}(x). \quad (1.2)$$

$\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  in (1.1) are the usual Pauli matrices and  $\sigma_4 = iI$  where  $I$  is the identity matrix. This theorem has important physical consequences. It provides an explanation for the baryon number violation in the standard model [2] and also for the relatively large value of  $\eta'$  in QCD [2,3]. An explicit computation of baryon number violating processes or of the  $\eta'$  mass needs a non-perturbative technique and the most promising method is the lattice formalism of gauge theories coupled to fermions. An investigation of the index theorem on the lattice is a necessary precursor to either of the above computations.

In order to investigate the extent to which the index theorem persists on the lattice, one needs good definitions for the topological charge of gauge fields and for the index of the chiral Dirac operator on the lattice. Unlike the continuum, there is no division of lattice gauge fields into disconnected classes since lattice gauge fields are characterized by group elements on the links of the lattice and all link variables can be deformed to unity. Of course, on a lattice there is no concept of smoothness associated with continuum gauge fields and therefore the previous statement is not a surprise. Yet, it is possible to associate an integer with a gauge field configuration on the lattice in such a way that it matches with (1.2) for continuum configurations [4]. Instead of using the geometrically inspired definition on the lattice [4], one could write down an expression for the integrand in (1.2) on each site in the lattice and simply sum this up over all lattice sites to get an expression for  $Q$ . Such a definition, referred to as a field theoretic definition, does not result in an integer value for  $Q$  but the probability distribution,  $p(Q)$ , is expected to have sharp peaks around integers close to the continuum limit. Even if this is the case, one expects difficulties with both the geometric definition and field theoretic definition due to short distance fluctuations.  $Q$  measures the topology of the gauge field configuration and is a globally defined quantity. But in both the definitions,  $Q$  is obtained as a certain sum over sites on the lattice and therefore large fluctuations of the order of lattice spacing can affect the measurement of  $p(Q)$ . This problem has plagued the measurement of  $p(Q)$

for almost a decade. One approach to resolve this problem, referred to as the “cooling method”, has been to first smooth out the gauge field configuration and then measure the topological charge using the field theoretic method [5]. In principle one could measure the charge using the geometric definition after cooling but it is not expected to be different from the field theoretic definition since the field configuration is smooth. The difficulty with the “cooling method” is the inability to precisely define the stopping criteria for the cooling procedure. This is due to the fact that the Wilson action on the lattice has only one minimum corresponding to zero field density. This difficulty was recently circumvented by an “improved cooling algorithm” [6] based on an improved action [7] designed to stabilize large instantons on the lattice. Recent results based on “improved cooling” [8] seem to indicate that this is a good procedure to measure the topological charge on the lattice. An approach that complements the cooling method is to measure directly using the generated lattice gauge field configurations but improve the field theoretic operators and also determine the non-perturbative renormalization of the operator in a systematic manner [9]. Since [8] deals with SU(2) theory and [9] deals with SU(3) theory a direct comparison is not possible. Both [8] and [9] deal with the measurement of the topological charge on the lattice using the standard single plaquette Wilson action. A different approach to circumvent the problem associated with short distance fluctuations is to improve the action and measure the topological charge without having to use any “cooling” technique [10]. Such a scheme does not seem to completely remove the problem of short distance fluctuations and this is resolved in [10] by interpolating the gauge fields to a finer lattice and using the geometric method to measure the topological charge on the fine lattice. It is possible to compare the result in [8] with the result in [10] but one has to assume universality and also scaling. The assumption of universality is needed in the sense that a large class of actions yield the same continuum limit for the distribution for the topological charge. The assumption of scaling is needed since only the continuum number in [8] and [10] can be compared. If one assumes universality and scaling then the number for the topological susceptibility,  $\chi$ , obtained in [8] is two to three standard deviations smaller than the one obtained in [10].

To investigate the remnant of the index theorem on the lattice and to compare with the different methods used to measure the topological charge using (1.2) one has to define the index of the chiral Dirac operator on the lattice. The obvious problem is the difficulty in formulating chiral fermions on the lattice. Smit and Vink carefully dealt with the definition of the index of  $C(A)$  on the lattice [11] by looking at the low lying eigenvalues of the lattice Dirac operator and the associated chiralities. They performed measurements in two dimensional U(1) theory using both Wilson [12] and staggered fermions [13] and in

four dimensional  $SU(3)$  theory using staggered fermions [14]. Assuming the index theorem, the topological susceptibility,  $\chi$ , in  $SU(3)$  theory was computed on the lattice [15] and compared with the various results for  $\chi$  using (1.2). The comparison revealed that the value for  $\chi$  based on the geometric method [16] was too large and the one obtained from “cooling” [5] was not as large but still larger than the one obtained using the fermionic method [15]. The fermionic method is inherently global in nature since eigenvalues of the Dirac operator are functions of the complete set of gauge fields on the lattice. In spite of this advantage, there is still a hurdle that one had to face when using Wilson fermions or staggered fermions to compute the index of  $C(A)$ . This is because one does not have exact zero modes on the lattice in either case. Wilson fermions do not have exact zero modes due to the Wilson term that breaks the chirality and one has to tune the Wilson parameter per gauge configuration to get a good estimate of the number of zero modes [12]. The lack of flavor symmetry in staggered fermions causes a zero mode shift [17] and one has to estimate the number of zero modes by looking at low lying eigenvalues [13,14]. Owing to this difficulty in estimating the zero modes in a precise manner, no definite conclusion could be reached from the comparison in [15]. A comparison of the estimate of  $\chi$  in [15] obtained using the fermionic zero modes is still smaller than the recent estimate of  $\chi$  in [9]. This difference could be due to improper estimate of zero modes in [15] or due to the an improper estimate of the non-perturbative renormalization in [9]. On the other hand, if we assume that both [9] and [15] are correct then the index theorem does not seem to be valid on the lattice and this would be a serious drawback for the lattice formulation.

The overlap formalism [18] developed to deal with chiral gauge theories on the lattice provides a natural framework to measure the index of the chiral Dirac operator as first realized in [19]. In this paper we present the first measurement of the distribution of the index of the chiral Dirac operator in a four dimensional gauge theory using the overlap formalism. The simplest theory from the numerical viewpoint is pure  $SU(2)$  gauge theory. Since this is the first measurement of this kind, we decided to use the standard single plaquette Wilson action and decided to investigate the continuum index theorem on the lattice by a direct comparison with one of the measurements in [8]. Therefore we chose a value of  $\beta = 2.4$  on a  $12^4$  lattice. This way we do not have to invoke universality or scaling for the sake of testing the continuum index theorem on the lattice. In section II we review the definition of the index of the chiral Dirac operator using the overlap formalism [18]. The overlap formalism has a parameter  $m$  which has to be held fixed at some value in the range  $0 < m < 1$  as one takes the continuum limit. In section III, we discuss the role played by this parameter in defining the index of the chiral Dirac operator. We discuss the numerical technique and results in section IV. We measured the

index,  $Q$ , on 140 independent configurations and obtained an estimate for the probability distribution,  $p(Q)$ . This distribution is in good agreement with a recent measurement [8] of the distribution of the topological charge on the same lattice using the same coupling and the same lattice gauge action. In particular we find  $\langle Q^2 \rangle = 3.3(4)$  to be compared with  $\langle Q^2 \rangle = 3.9(5)$  found in [8]. The good agreement between the two distributions is an indication that the continuum index theorem can be carried over in a probabilistic sense on to the lattice. We present our conclusions and directions for future work in section V.

## II. Index of $C(A)$ in the overlap formalism

The index of the chiral Dirac operator,  $C(A)$ , is the difference between the kernel of  $C(A)$  and the kernel of  $C^\dagger(A)$ . If the two kernels are not the same then the operator  $C(A)$  is a map between two spaces that differ in dimension. Formally, in terms of the unregulated fermionic action for a single left handed chiral fermion,

$$S_f = \int d^4x \bar{\psi}_l(x) C(A) \psi_l(x) \quad (2.1)$$

it means that the “number” of  $\bar{\psi}_l$  degrees of freedom is different from the “number” of  $\psi_l$  degrees of freedom. Therefore the path integral over all  $\bar{\psi}_l$  and  $\psi_l$  yields a zero unless one inserts the excess degrees of freedom of  $\bar{\psi}_l$  or  $\psi_l$  inside the path integral. Since one has to insert a certain number of  $\bar{\psi}_l$  or  $\psi_l$  as the case may be, one gets a non-zero expectation value for a fermion operator where the operator does not have an equal number of  $\bar{\psi}_l$  and  $\psi_l$ . This results in the existence of ’t Hooft processes [2] in chiral gauge theories (eg: Weak interactions in the Standard Model) and vector gauge theories (eg: QCD).

The overlap formalism [18] provides a formula for the generating functional associated with the path integral of chiral fermions in a fixed gauge background. It has the important property of respecting the index of  $C(A)$ . The overlap formula for the generating functional for a left handed chiral fermion coupled to an external gauge field,  $U$ , on the lattice is

$$Z(\bar{\eta}, \eta, U) = {}^{WB}_U \langle L - | e^{\bar{\eta}a + \eta a^\dagger} | L + \rangle_U^{WB} \quad (2.2)$$

where  $|L \pm \rangle_U^{WB}$  are many body ground states\* of

$$\mathcal{H}^\pm = a^\dagger \mathcal{H}^\pm(U) a \quad (2.3)$$

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\* The superscript, WB, implies a certain phase choice needed to fully define the states entering in the generating functional. Since the phase of the overlap does not enter the definition of the index, the details of the phase choice will not be of concern to us in this paper.

$$H^\pm(U) = \begin{pmatrix} B(U) \mp m & C(U) \\ C^\dagger(U) & -B(U) \pm m \end{pmatrix} \quad (2.4)$$

$$C(x\alpha i, y\beta j; U) = \frac{1}{2} \sum_{\mu=1}^4 \sigma_\mu^{\alpha\beta} \left[ \delta_{y,x+\hat{\mu}} (U_\mu(x))^{ij} - \delta_{x,y+\hat{\mu}} (U_\mu^\dagger(y))^{ij} \right] \quad (2.5)$$

$$B(x\alpha i, y\beta j; U) = \frac{1}{2} \delta_{\alpha\beta} \sum_{\mu=1}^4 \left[ 2\delta_{xy} \delta_{ij} - \delta_{y,x+\hat{\mu}} (U_\mu(x))^{ij} - \delta_{x,y+\hat{\mu}} (U_\mu^\dagger(y))^{ij} \right] \quad (2.6)$$

$0 < m < 1$  in (2.4) and all values of  $m$  in this range correspond to the same physical theory in the continuum limit. Different choices of  $m$  result in different finite lattice spacing effects.  $B$  is hermitian and so are the Hamiltonians  $H^\pm(U)$ . On a finite lattice, the single particle Hamiltonians,  $H^\pm(U)$ , are finite matrices and the many body states,  $|L \pm\rangle_U^{\text{WB}}$ , are obtained by filling all the negative energy states of  $H^\pm(U)$ . If the number of negative energy states of  $H^-(U)$  are different from that of  $H^+(U)$  then the number of bodies making up  $|L-\rangle_U^{\text{WB}}$  is different from the number that makes up  $|L+\rangle_U^{\text{WB}}$ . We need to differentiate a certain number of times with respect to  $\eta$  or  $\bar{\eta}$  in (2.2) to compensate for this difference in the number of bodies and get a non-zero result. This difference in the number of bodies is the index of  $C(U)$ .

To understand the connection between the spectrum of  $H^\pm(U)$  and index of  $C(U)$  and also to develop an efficient numerical algorithm to measure the index using the above connection, it is useful to consider the eigenvalue flow of

$$H(\mu) = \begin{pmatrix} B - \mu & C \\ C^\dagger & -B + \mu \end{pmatrix} \quad (2.7)$$

as a function of  $\mu$  in a fixed gauge background  $U$  (we have suppressed the dependence on  $U$  in (2.7)). Our interest is in comparing the spectrum at  $\mu = -m$  ( $H^- = H(-m)$ ) with the spectrum at  $\mu = m$  ( $H^+ = H(m)$ ). One can prove that  $H(\mu)$  has an equal number of positive and negative eigenvalues for all  $\mu < 0$  [18,20]. To see this, we first note that  $u^\dagger B u \geq 0$  for all vectors  $u$  and consider the possibility of a zero eigenvalue for  $H(\mu)$  with eigenvector  $\begin{pmatrix} u \\ v \end{pmatrix}$ :

$$\begin{aligned} (B - \mu)u + Cv &= 0; & C^\dagger u + (\mu - B)v &= 0; & u^\dagger u + v^\dagger v &= 1 \\ \Rightarrow u^\dagger [(B - \mu)u + Cv] - \left( v^\dagger [C^\dagger u + (\mu - B)v] \right)^* &= 0 \\ \Rightarrow u^\dagger B u + v^\dagger B v &= \mu \end{aligned} \quad (2.8)$$

The above equation can have a solution only if  $\mu \geq 0$ . Since the spectrum has an equal number of positive and negative eigenvalues for  $\mu = -\infty$ , it follows that the spectrum has an equal number of positive and negative eigenvalues for all  $\mu < 0$ .

For  $\mu \geq 0$ ,  $H(\mu)$  can have zero eigenvalues and if this is the case for  $\mu < 1$ , then the spectral flow of  $H(\mu)$  will have some level crossings. Such level crossings are expected to be of generic type in that the first derivative of the crossing eigenvalue is not zero. Let  $n^+$  denote the number of crossings with positive slope (a negative eigenvalue becoming positive as  $\mu$  is increased) and let  $n^-$  denote the number of crossings with negative slope. If  $n^+ \neq n^-$ , the spectrum at  $\mu = 1$  does not have an equal number of positive and negative eigenvalues implying that the index of  $C(U)$  for this particular  $U$  is equal to  $Q = n^+ - n^-$ . Given a configuration  $U$  with index  $Q$  on a finite lattice, it can be shown that the parity transformed partner of  $U$  has index  $-Q$ . This is a consequence of Lemma 4.4 in [18]. This proves that the distribution of  $Q$ , namely  $p(Q)$  is symmetric about  $Q = 0$ .

The slope of flow at the crossing from first order perturbation theory of  $H(\mu)$  with respect to  $\mu$  is

$$\frac{d\lambda}{d\mu} = -(u^\dagger u - v^\dagger v) \quad (2.9)$$

where  $\begin{pmatrix} u \\ v \end{pmatrix}$  is the eigenvector with eigenvalue  $\lambda$  at  $\mu$ . Since

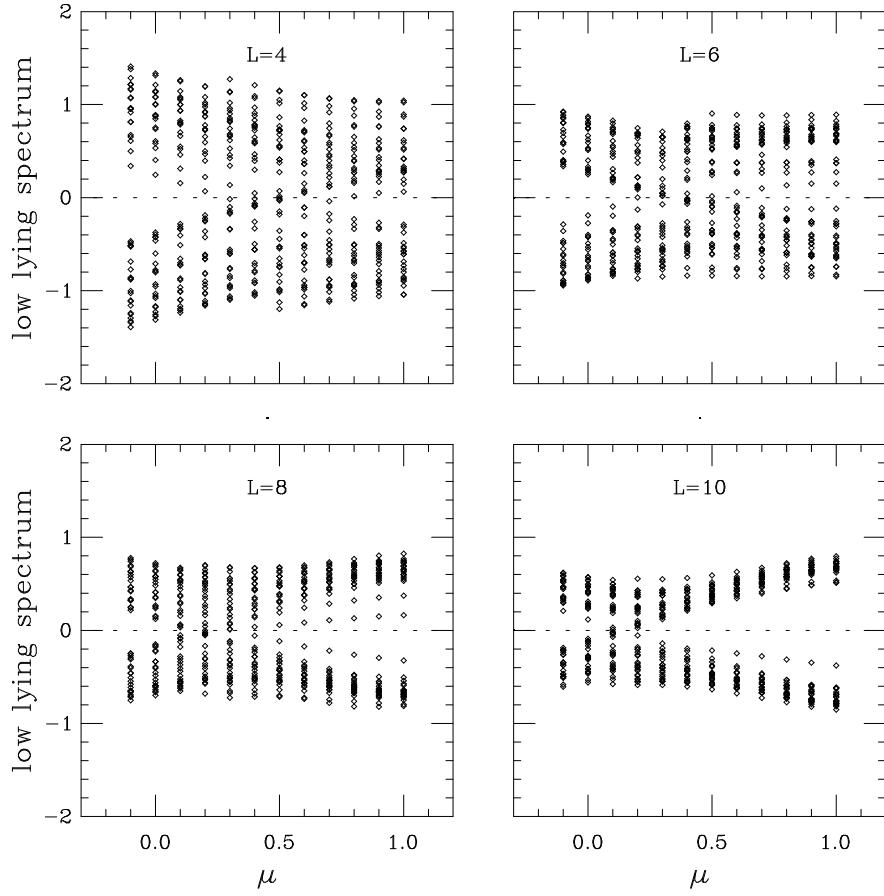
$$u^\dagger u - v^\dagger v = (u^\dagger \quad v^\dagger) \gamma_5 \begin{pmatrix} u \\ v \end{pmatrix}; \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.10)$$

a positive slope can be associated with one chirality and a negative slope with the opposite chirality. Such an association leads to a connection with the continuum definition of the index of  $C(U)$  as was shown in some detail in [18].

This shows that one can obtain the index  $Q$  of  $C(U)$  by looking at the spectral flow of  $H(\mu)$ , measuring  $n^\pm$ , and computing  $n^+ - n^-$ . (2.9) and (2.10) imply that one can associate a chirality with each crossing. Therefore, we can speculate that the gauge field configuration  $U$  is made up of  $n^+$  instanton like objects and  $n^-$  anti-instanton like objects based on the level crossings. If  $n^+ = n^-$  then the overlap of the two many body states need not be zero and hence a non-zero number for  $n^+$  in this case does not necessarily mean the existence of zero modes.

### III. The role of $m$

For a given gauge field configuration on a finite lattice away from the continuum, any crossing will occur at some  $\mu > 0$  and if there are more than one crossings they will generically occur for different values of  $\mu$ . As one goes to the continuum, these crossings will occur for smaller values of  $\mu$  and approach  $\mu = 0$  in the continuum limit indicating a topologically non-trivial gauge field configuration. This can be seen from equation 2.8.



**Figure 1** Spectrum of  $H(\mu)$  as a function of  $\mu$  in the two dimensional  $U(1)$  model at a fixed physical volume of  $\frac{gL}{\sqrt{\pi}} = 3.0$  on four different  $L \times L$  lattices corresponding to four different lattice spacings proportional to  $\frac{1}{L}$ . For each lattice the three low lying positive and negative eigenvalues of eight different gauge field configurations are shown.

Close to the continuum  $U_\mu(x) \sim 1 \Rightarrow B(x, y; U) \ll 1 \Rightarrow u^\dagger B u + v^\dagger B v = \mu \ll 1$ . To understand the crossing region in  $\mu$  as we approach the continuum limit, we study the spectrum as a function of  $\mu$  in the model with  $U(1)$  gauge fields on a two dimensional torus [21]. The gauge field action is the standard Wilson plaquette action. We fix the physical volume,  $l^2$ , by choosing a fixed value for  $\frac{el}{\sqrt{\pi}}$  where  $e$  is the gauge coupling. To work in a constant physical volume on the lattice, we impose  $\frac{el}{\sqrt{\pi}} = \frac{gL}{\sqrt{\pi}}$  on an  $L \times L$  lattice with  $g$  being the coupling in lattice units. The lattice spacing is proportional to  $\frac{1}{L}$ . Fixing  $\frac{gL}{\sqrt{\pi}} = 3.0$ , we generated several gauge configurations on  $L = 4, 6, 8, 10$ . In Figure 1, we plot the spectrum of several configurations where there were level crossings. As one can

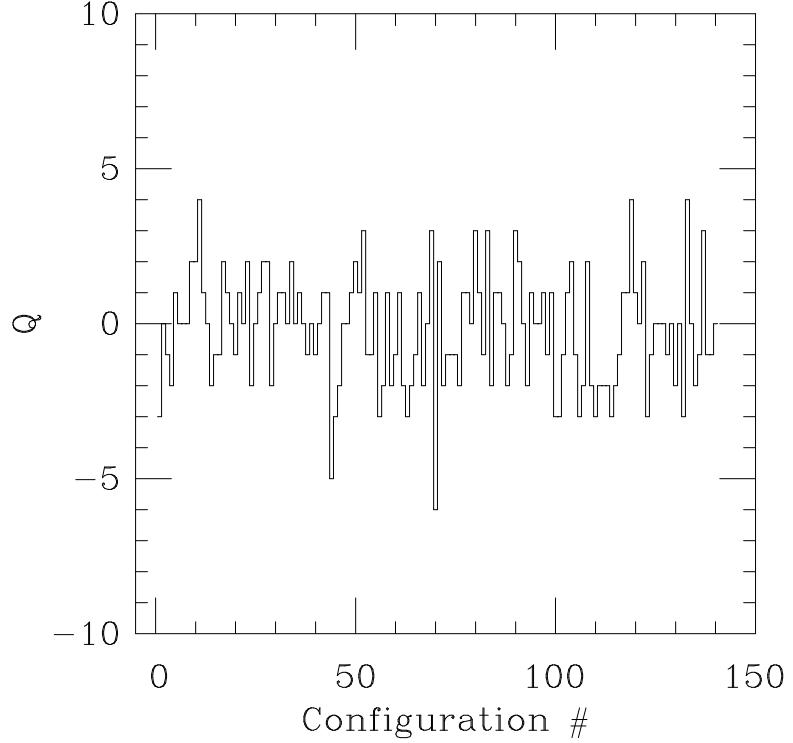
readily see, the crossings happen in a small range of  $\mu > 0$  and this range gets closer to  $\mu = 0$  as  $L$  gets bigger.

To define the number of crossings and to measure the probability distribution of the index,  $p(Q)$ , on a finite lattice we will need to fix a value of  $\mu$  and count only crossings below that  $\mu$ . As it is clear from the above discussion the actual value of  $\mu$  that we pick ( $0 < \mu < 1$ ) will become unimportant as the continuum limit is approached since then all crossings will occur in the neighborhood of 0. However, the approach of  $Q$  toward the continuum limit will clearly depend on  $\mu$ . For example, from figure 1 we see that if we set  $\mu = 0.4$  then the  $L = 4$  case will have almost no crossings below  $\mu$  indicating a severe finite lattice spacing effect. The  $L = 6$  case will have some of its crossings below  $\mu$  and the  $L = 8, L = 10$  cases will have all of their crossings below  $\mu$ . On the other hand if we set  $\mu \sim 1$  all four cases have their crossings below  $\mu$  and the continuum limit is approached in a smoother way. It is therefore advantageous to choose  $\mu \sim 1$  especially since in most cases one is forced to work far away from the continuum limit. Since most crossings happen within a small interval of  $\mu$  in practice one scans the full  $\mu \in (0, 1)$  interval using a large step size while concentrating in the region where most crossings occur using a finer step size. For our study of the SU(2) lattice gauge theory in four dimensions we consider all crossings below  $\mu = 1$ .

#### IV. Numerical technique and results

In this section we present results for  $p(Q)$  obtained by a direct measurement of  $Q$ , the index of  $C(U)$ , using the overlap formalism in pure SU(2) gauge theory. We use the exponentiated standard Wilson single plaquette action as the Boltzmann weight. We have used the standard heat-bath technique to generate the gauge field configurations [22]. Having generated the gauge field configuration, the index associated with that configuration is measured by obtaining the flow of  $H(\mu)$  defined in (2.7).

On an  $L^4$  lattice, the hamiltonian is an  $8L^4 \times 8L^4$  matrix. Like in the two dimensional U(1) model shown in Figure 1, the spectrum for the four dimensional SU(2) theory also has two bands: one is a band of positive eigenvalues and one is a band of negative eigenvalues. Each band has two edges and the spectrum has four edges. The Lanczos algorithm is designed to efficiently extract the edges of a spectrum and we used the standard Lanczos algorithm [23] to get the eigenvalues on these four edges. The part of the spectral flow we are interested in is associated with the two inner edges, namely the two edges close to zero. We fix a value of  $\mu$  and perform sufficient Lanczos iterations to obtain at least five low lying positive and five low lying negative eigenvalues to a sufficient accuracy. We then do the same for several values of  $\mu$ . For each value of  $\mu$  we calculate at two points close to



**Figure 2** Plot of the index,  $Q(i)$ , versus the configuration number,  $i$ , as obtained in our numerical simulation.

each other namely  $\mu \pm 0.0025$ . This gives the slope of the flow lines at each  $\mu$  and makes the job of identifying the flow lines considerably easier. All our crossings happen in the range of  $0.7 < \mu < 1.0$  and so we chose the following set of values for  $\mu$ :

$$\begin{aligned} \mu = \{ & -0.1, 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, \\ & 0.73, 0.76, 0.79, 0.82, 0.85, 0.88, 0.91, 0.94, 0.97, 1.0 \} \end{aligned}$$

This enables us to perform a careful study of the spectral flow and estimate the number of  $n^+$  and  $n^-$  per gauge configuration.

Starting from a random configuration we performed 5000 heat bath updates of the lattice to achieve thermalization. We then measured the index of 140 gauge field configurations where two adjacent configurations were separated by 100 heat bath updates of the lattice. Having generated a gauge field configuration, its parity transformed partner has the same action and therefore a parity transformation will always be accepted by any

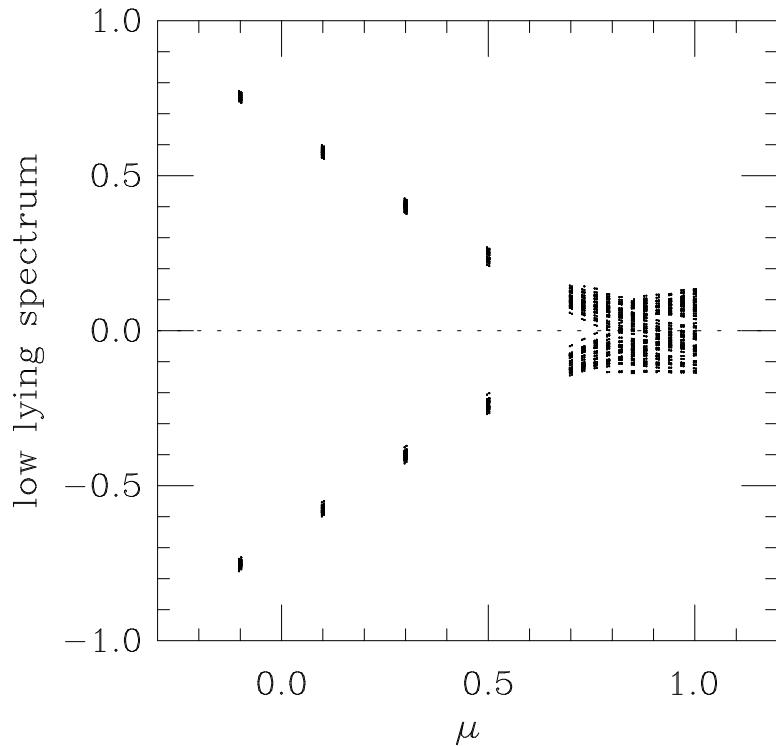
updating scheme. We assume that this is done for every configuration and therefore the distribution,  $p(Q)$ , is forced to be symmetric. Our data consists of 140 configurations and their parity transformed partner. In any error analysis, we assume that we have only 140 independent configurations.

A plot of  $Q$  as a function of the configuration number is shown in figure 2. For this plot we have ignored configurations obtained by a parity transformation and is therefore a plot of raw data. There is no evidence for any auto-correlation in this plot. In figure 3 we plot an ensemble of the spectral flow that we obtained. There are a total of five configurations in this ensemble and depicts the typical spectral flow. The flow is very similar to the ones obtained in figure 1 for the  $U(1)$  model in two dimensions. In this case, the crossings occur in the region  $0.7 < \mu < 1.0$ . Figure 4 focuses on  $0.7 \leq \mu \leq 1.0$  where crossings occur and show four typical configurations with different values of  $Q$ . For each configuration the number of crossings was deduced by visually inspecting figures similar to the ones shown in figure 4. Since our ensemble contains only 140 configurations this was not a time consuming task. However, for much larger ensembles the process should be automated. This will require some form of pattern recognition and presents an interesting programming problem.

$Q$	$p(Q)$	$p(Q)[8]$
0	0.214(50)	0.218(37)
$\pm 1$	0.186(20)	0.168(31)
$\pm 2$	0.129(19)	0.122(23)
$\pm 3$	0.061(14)	0.067(25)
$\pm 4$	0.011(6)	0.025(13)
$\pm 5$	0.004(3)	0.005(4)
$\pm 6$	0.004(3)	0.005(5)
$\langle Q^2 \rangle$	3.3(4)	3.9(5)

**Table 1:** Comparison of the probability distribution of the index of  $C(U)$  obtained here with the probability distribution of the topological charge obtained in [8].

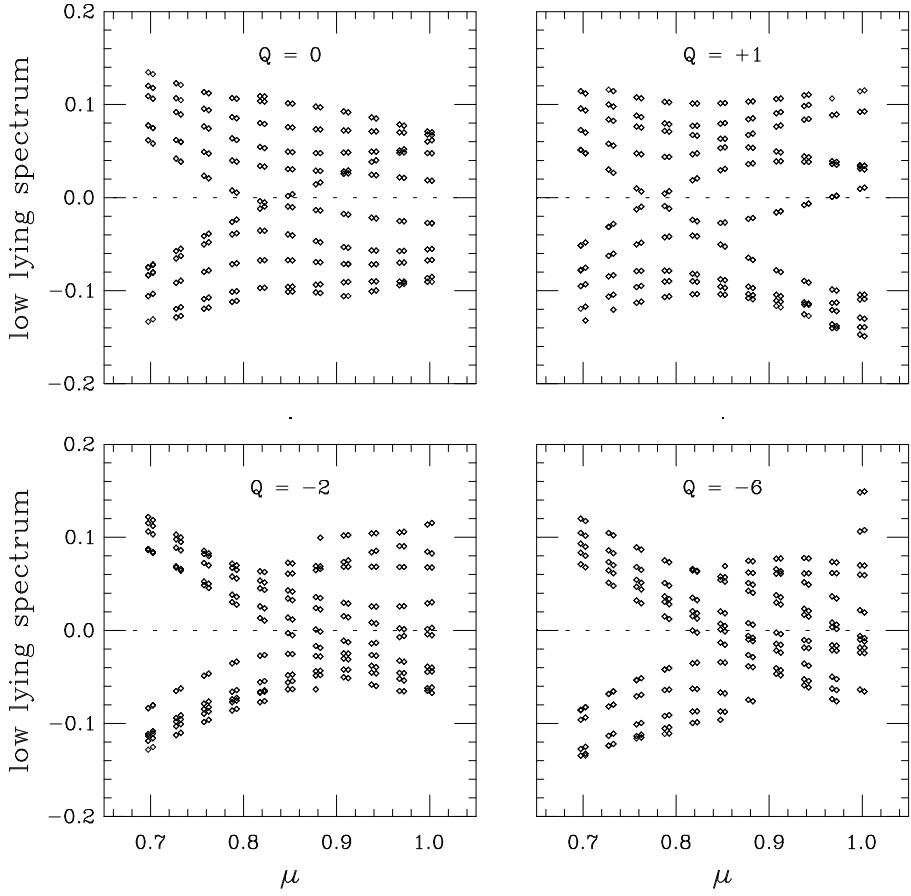
Since there is data for the distribution of topological charge using improved cooling for the same lattice parameters ( $\beta = 2.4$  on a  $12^4$  lattice) using the same Wilson single plaquette action a direct comparison of the two sides of the continuum index theorem on the lattice is now possible. This comparison is shown in table 1 and figure 5. The second column in table 1 is our result for the symmetrized distribution,  $p(Q)$  where  $Q$  is



**Figure 3** Five low lying positive and negative eigenvalues of  $H(\mu)$  as a function of  $\mu$  for five different gauge field configuration. The configurations are from pure  $SU(2)$  gauge theory at  $\beta = 2.4$  on a  $12^4$  lattice.

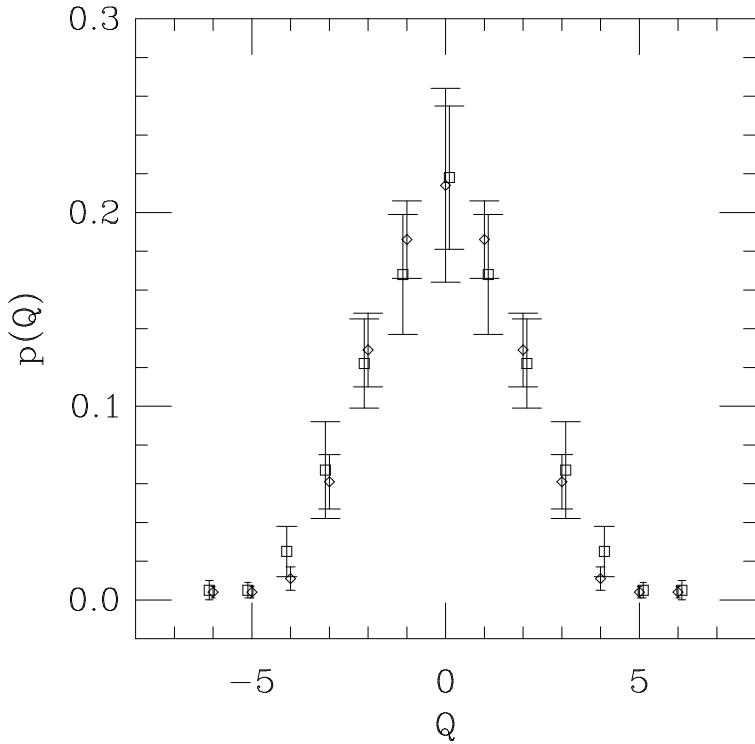
the index of  $C(U)$ . The third column in table 1 is the result from [8] for the symmetrized distribution,  $p(Q)$  where  $Q$  is the topological charge given by (1.2). The two columns are a result of measurements on a different set of independent configurations. The close matching of the two distributions indicate that the connection between the index of  $C(U)$  and the topological charge of  $Q(U)$  remains valid on the lattice in a probabilistic sense. The variance of  $Q$ , namely  $\langle Q^2 \rangle$ , obtained here and in [8] is also listed in Table 1 and they agree quite well. If we use a lattice spacing of  $a = 0.12\text{fm}$  associated with  $\beta = 2.4$ , then we get  $\chi = 184(6)\text{MeV}$ . To properly verify this as a continuum result we will have to show evidence for scaling in  $p(Q)$  where  $Q$  is the index of  $C(U)$ .

Now we turn our attention to  $p(n^+, n^-)$ , the probability of finding a configuration with  $n^+$  instantons like objects and  $n^-$  anti-instanton like objects. Such a configuration is



**Figure 4** Five low lying positive and negative eigenvalues of  $H(\mu)$  as a function of  $\mu$  for four gauge field configurations with different values of  $Q$ . The configurations are from pure  $SU(2)$  gauge theory at  $\beta = 2.4$  on a  $12^4$  lattice .

expected to have  $(n^+ + n^-)$  localized objects and the Dirac operator is expected to have the same number of localized eigenmodes. We assume that every crossing in a typical flow like the ones in figure 4 is associated with a localized eigenmode of the Dirac operator. By looking at the number of crossings with positive slope and number of crossings with negative slope, we obtain the  $n^+$  and  $n^-$  associated with that gauge configuration. Given a configuration with a fixed  $n^+$  and  $n^-$  we know that the parity transformed gauge field will have  $n^+$  and  $n^-$  interchanged. Therefore  $p(n^+, n^-) = p(n^-, n^+)$  and we have listed  $p(n^+, n^-)$  for  $n^+ \leq n^-$  in Table 2 as obtained from the 140 independent measurements. In our sample we did not find any configuration with more than six objects. From the distribution we obtain  $\langle (n^+ + n^-) \rangle = 2.3(1)$  implying that on an average the gauge field



**Figure 5** Comparison of the probability distribution of the index of  $C(U)$  obtained here (diamonds) with the probability distribution of the topological charge obtained in [8] (squares). The squares have been slightly shifted laterally for visual purposes.

configuration has roughly two to three localized objects. We have obtained this number by simply taking the gauge field configuration without applying any cooling or smoothing algorithm. A similar result has also been obtained in [8] but their result is a function of the cooling sweeps. It is interesting to note that our result for  $\langle (n^+ + n^-) \rangle$  is compatible with the result obtained in [8] after 150 cooling sweeps. It would be interesting to obtain the scaling form of  $p(n^+, n^-)$  using the method described here and compare it with a similar result using the method in [8]. A natural assumption is that the result for  $p(n^+, n^-)$  in [8] would be unaffected by cooling if the lattice coupling is well within the scaling region and would be in agreement with the result obtained using the overlap formalism.  $p(n^+, n^-)$  has also information about interactions between instantons and anti-instantons.

Let us assume that all instantons are identical and all anti-instantons are identical. A model of non-interacting objects is consistent with our data. For example, we see that  $2p(0, 2) = p(1, 1)$  within errors indicating no difference in interactions between like and unlike objects. Also we see that  $2p^2(0, 2) = 3p(0, 1)p(0, 3)$  within errors indicating that there is no interaction between like objects. Of course the errors in the data in table 2 are too large to make any definite conclusion or help in building an empirical model for the interaction of instantons and anti-instantons. We see that the zero topological sector is dominated by configurations with  $n^+ \neq 0$  and this is thought to be relevant for a study of chiral symmetry breaking [24]. A better estimate of  $p(n^+, n^-)$  would facilitate a comparison with existing models for instanton–anti-instanton interactions [24].

$n^+$	$n^-$	$p(n^+, n^-)$
0	0	0.036(16)
0	1	0.114(15)
0	2	0.104(19)
0	3	0.057(13)
0	4	0.007(5)
0	5	0.004(3)
0	6	0.004(3)
1	1	0.164(39)
1	2	0.057(12)
1	3	0.025(12)
1	4	0.004(3)
1	5	0.004(3)
2	2	0.014(10)
2	3	0.014(6)

**Table 2:** Probability distribution of configurations with  $n^+$  instanton like objects and  $n^-$  anti-instanton like objects. This distribution yields  $\langle(n^+ + n^-)\rangle = 2.3(1)$ .

## V. Conclusions

In this paper we have used the overlap formalism [18] developed to deal with determinants of chiral Dirac operators as a tool to measure the Atiyah-Singer index of lattice gauge fields. By working with the standard single plaquette action on a  $12^4$  lattice with a coupling of  $\beta = 2.4$  we could directly compare the probability distribution of  $Q$ , the index of  $C(U)$ , obtained here with the probability distribution of the topological charge obtained in [8]. Table 1 provides this comparison. The good agreement between the two

distributions is an indication that the continuum index theorem can be carried over in a probabilistic sense on to the lattice.

In this paper we have only done a single coupling on a single lattice. More such measurements have to be done to establish scaling of the probability distribution of  $Q$  and follow the index theorem to the continuum. The algorithm to measure the index involves a Lanczos procedure and the Hamiltonian operators involved are directly related to the usual Wilson-Dirac operators. Our code was written for a vector machine. Each measurement of  $Q$  on a single gauge configuration involved 2000 Lanczos iterations and took a total of 2.75 hours of CPU time on a single processor of the CRAY C90 machine. The time will scale linearly with the number of lattice points and linearly with the number of Lanczos iterations. It is possible to study scaling in a systematic manner with the use of present day computers. This work is currently in progress.

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